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Analysis of a nonlinear oscillator with discontinuity

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ABSTRACT

In this paper, a nonlinear oscillator with discontinuity is analyzed using He's amplitude–frequency formulation. The natural frequency of the oscillator system can be obtained using concise and straightforward calculation steps. Good agreement between the results from the proposed and other classical approaches is reached.

1. Introduction

In this paper, we consider the following nonlinear oscillator with discontinuity

$$\frac{d^2u}{dt^2} + u\,|u| = 0 \tag{1}$$

with initial conditions

$$u(0) = A, \qquad u'(0) = 0. \tag{2}$$

The research topic of nonlinear oscillator with discontinuity has been studied widely using various methods, for example, the homotopy perturbation method [1], the parameter-expansion method [2–4], the variational iteration method [5,6], and He's energy balance method [7,8]. A compete review on analytical methods is available on [9,10]. Recently, there have been some other researchers working on the analysis of nonlinear oscillator using ancient Chinese mathematics [11–19]. In this paper, we will apply the amplitude–frequency formulation [9], which was derived using the ancient Chinese mathematics in [11,12].

2. The amplitude–frequency formulation

The amplitude–frequency formulation was first proposed by He [9] in his review article in a few lines, and it was modified by Geng and Cai [20], and He [21,22], respectively. The formulation was used as a concise tool to find the relationship between the amplitude and frequency of a nonlinear oscillator [23–25]. This study furthers the previous works and aims to apply the formulation to analyzing a nonlinear oscillator with discontinuity.

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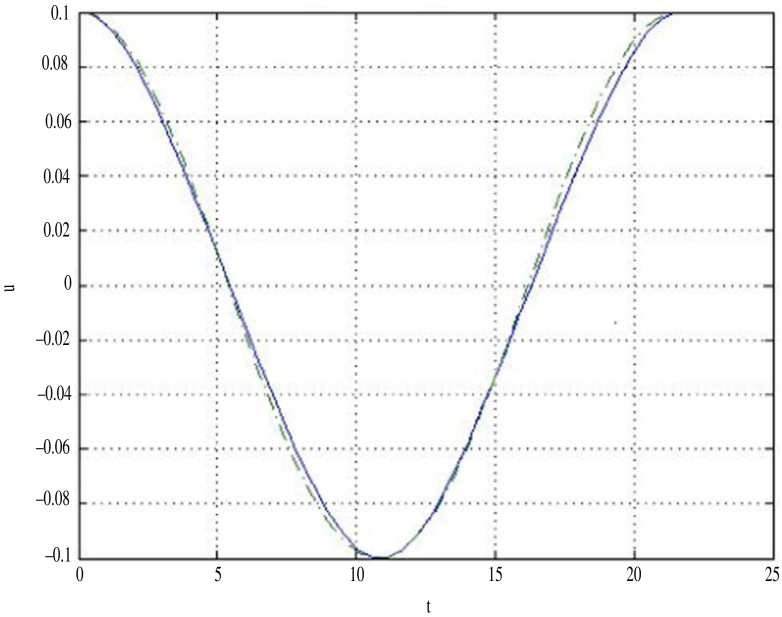
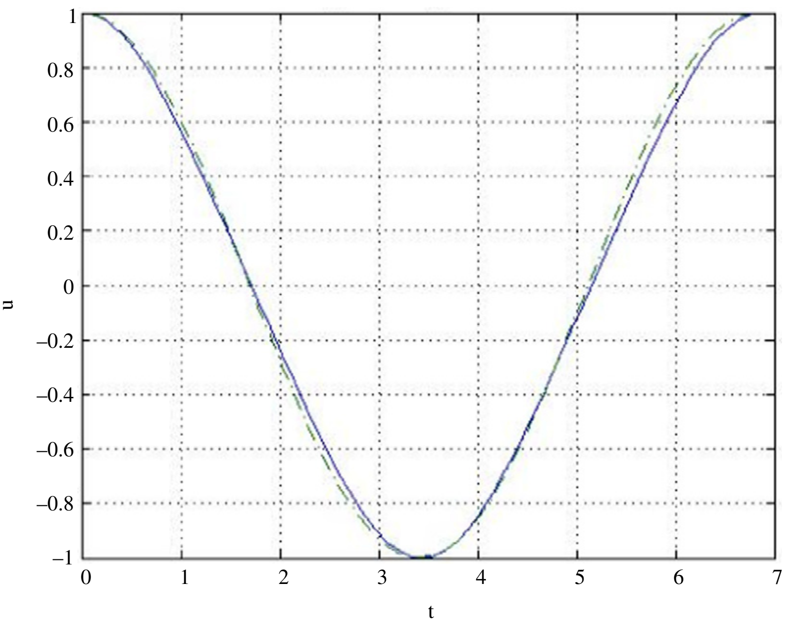


(a) $A = 0.1,\ \omega_{\text{app}} = 0.29135.$



(b) $A = 1,\ \omega_{\text{app}} = 0.92132.$

Fig. 1. Comparison between the proposed approximate solution and the exact solution. Dashed line: Approximate solution; continuous line: Exact solution.

According to He's amplitude–frequency formulation [9], the following two trial functions are selected

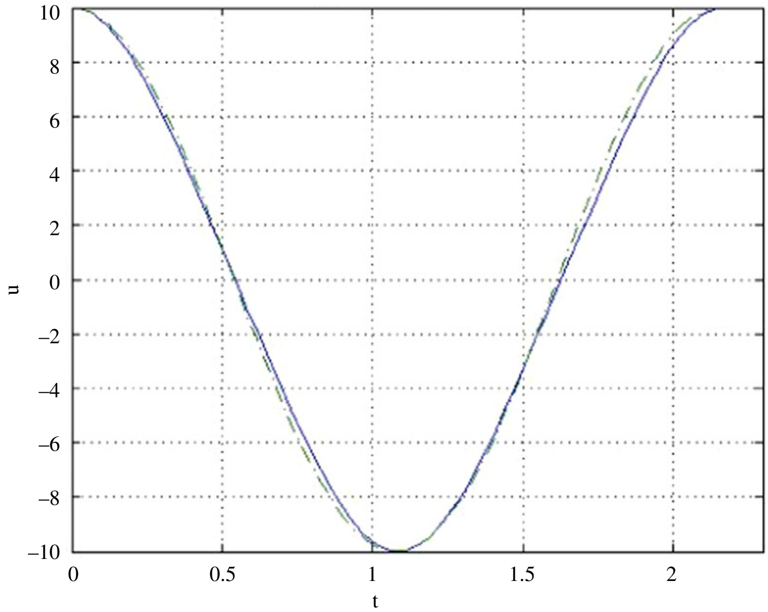
$$u_1(t) = A\cos t, \quad \text{and} \quad u_2(t) = A\cos\omega t.$$

They are the solutions of the linear differential equations in Eqs. (3)–(4), respectively,

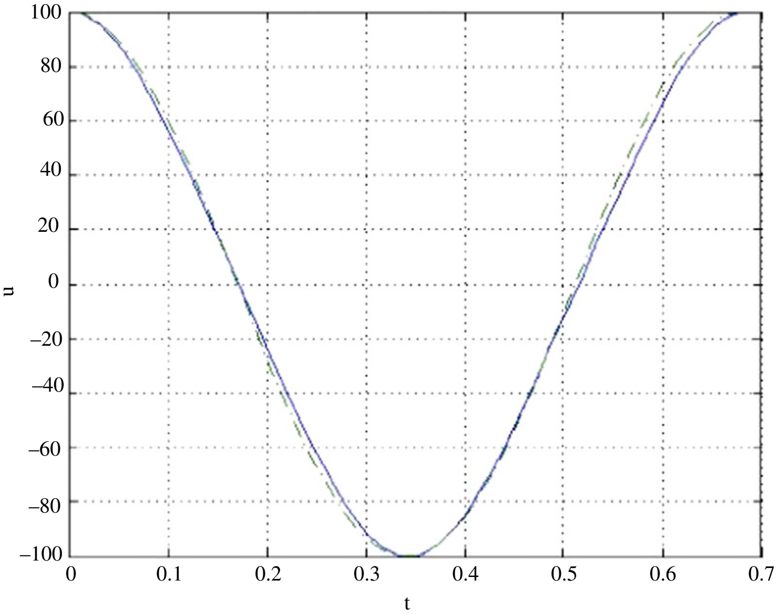
$$u'' + \omega_1^2 u = 0, \quad \omega_1^2 = 1 \tag{3}$$

$$u'' + \omega_2^2 u = 0, \quad \omega_2^2 = \omega^2, \tag{4}$$

where ω is assumed to be the natural frequency of the nonlinear oscillator in Eq. (1).



(c) $A = 10,\ \omega_{\text{app}} = 2.9135$.



(d) $A = 100,\ \omega_{\text{app}} = 9.2132$.

Fig. 1. (*continued*)

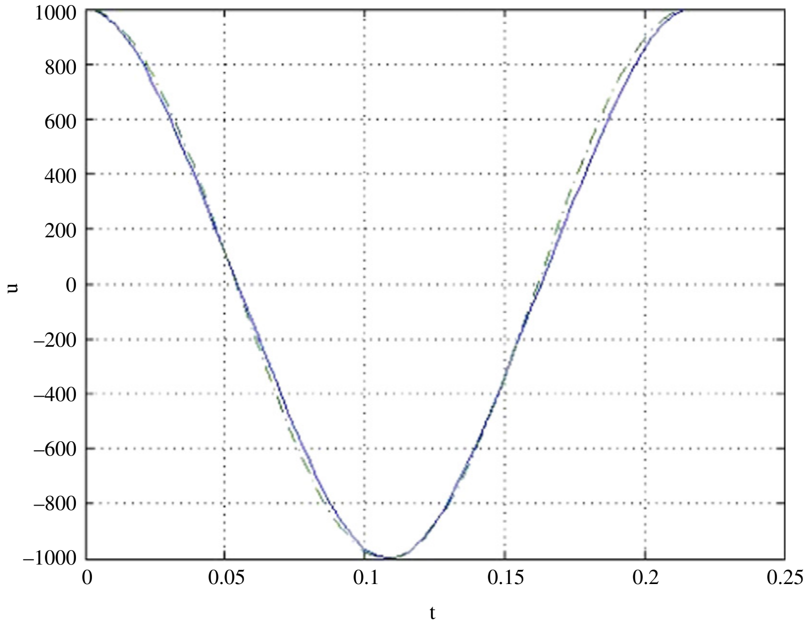
Hence, the residuals in Eq. (1) are

$$R_1(t) = -A\cos t + A\cos t\,|\cos t| \tag{5}$$

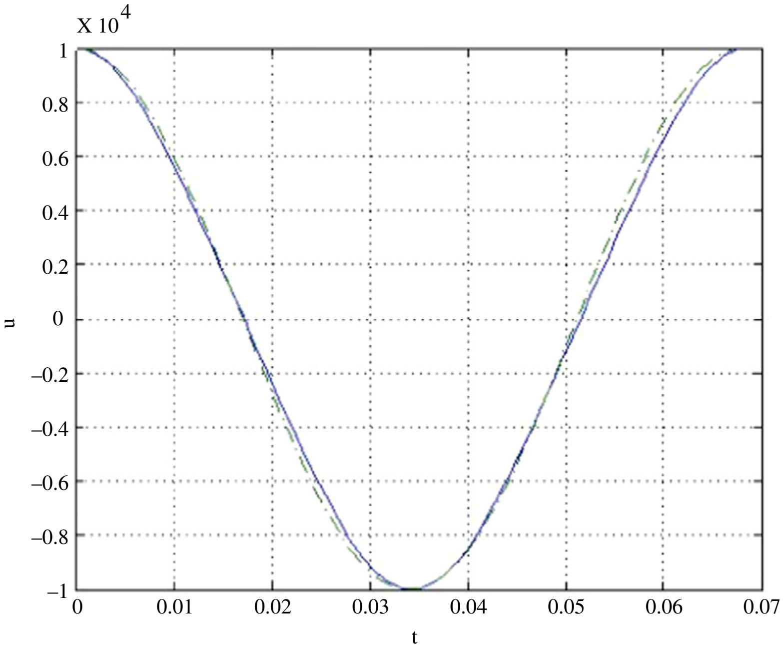
$$R_2(t) = -A\omega^2\cos\omega t + A\cos\omega t\,|\cos\omega t|\,. \tag{6}$$

Using He's amplitude–frequency formulation [9,21,22], we can approximately determine ω^2 in the form

$$\omega^2 = \frac{\omega_1^2\widetilde{R}_2 - \omega_2^2\widetilde{R}_1}{\widetilde{R}_2 - \widetilde{R}_1}, \tag{7}$$



(e) $A = 1000,\ \omega_{\text{app}} = 29.1346$.



(f) $A = 10\,000,\ \omega_{\text{app}} = 92.1318$.

Fig. 1. (*continued*)

where

$$\begin{aligned}\widetilde{R}_1 &= \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos\left(\frac{2\pi}{T_1}t\right)\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\pi/2}(-A\cos t + A^2\cos t\,|\cos t|)\cos(t)\mathrm{d}t\end{aligned}$$

$$= \left(\frac{4A}{3\pi} - \frac{1}{2}\right) A \tag{8}$$

and

$$\begin{aligned}
\widetilde{R}_2 &= \frac{4}{T_2}\int_0^{T_2/4} R_2(t)\cos\left(\frac{2\pi}{T_2}t\right)\mathrm{d}t \\
&= \frac{4}{T_2}\int_0^{T_2/4} (-A\omega^2\cos\omega t + A^2\cos\omega t\,|\cos\omega t|)\cos(\omega t)\mathrm{d}t \\
&= \frac{2}{\pi}\int_0^{\pi/2} (-A\omega^2\cos^2 s + A^2\cos^3 s)\mathrm{d}s \\
&= \left(\frac{4A}{3\pi} - \frac{\omega^2}{2}\right) A.
\end{aligned} \tag{9}$$

We, therefore, have

$$\omega^2 = \frac{\omega_1^2\widetilde{R}_2 - \omega_2^2\widetilde{R}_1}{\widetilde{R}_2 - \widetilde{R}_1} = \frac{\left(\frac{4A}{3\pi} - \frac{\omega^2}{2}\right)A - \omega^2\left(\frac{4A}{3\pi} - \frac{1}{2}\right)A}{\left(\frac{4A}{3\pi} - \frac{\omega^2}{2}\right)A - \left(\frac{4A}{3\pi} - \frac{1}{2}\right)A} = \frac{8A}{3\pi}. \tag{10}$$

Its approximate frequency reads

$$\omega = \sqrt{\frac{8A}{3\pi}}. \tag{11}$$

Its approximate solution is obtained as follows

$$u(t) = A\cos\sqrt{\frac{8A}{3\pi}}t. \tag{12}$$

Note that Eq. (12) is valid for $0 < A < +\infty$. Fig. 1 shows the comparisons between the approximate and exact displacements. Besides, according to [26], the exact natural frequency is given by

$$\omega_{\text{exact}} = \frac{\sqrt{6A}\cdot\pi}{2\mathrm{B}\left(\frac{1}{2},\frac{1}{3}\right)}, \tag{13}$$

where $\mathrm{B}(*, *)$ denotes the Beta function.

The relative error of Eq. (12) is given by

$$\frac{\omega_{\text{exact}} - \omega_{\text{app}}}{\omega_{\text{exact}}} = 1 - \frac{\sqrt{\frac{8A}{3\pi}}}{\frac{\sqrt{6A}\cdot\pi}{2\mathrm{B}\left(\frac{1}{2},\frac{1}{3}\right)}} = -0.0073. \tag{14}$$

Therefore, the accuracy is 0.73%.

3. Conclusion

He's frequency formulation has been applied to the analysis of a nonlinear oscillator with discontinuity. It can be seen in the comparisons with the exact solutions that the results from the proposed method achieve high accuracy. One of the main advantages of the proposed method is that the solution steps are concise and straightforward.

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